Bisimulation Games Played in Fibered Categories

Ichiro Hasuo

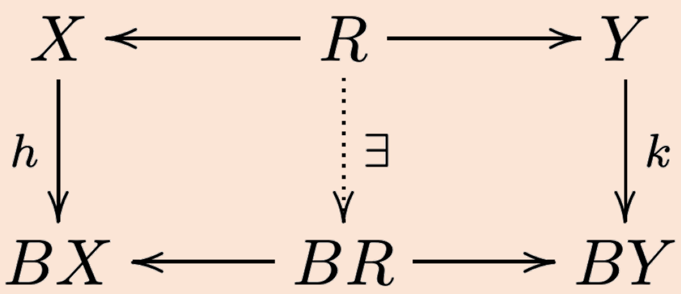
National Institute of Informatics (NII), Tokyo, Japan

Based on

- Komorida, Katsumata, Hu, Klin & Hasuo, LICS'19
- Komorida, Katsumata, Kupke, Rot & Hasuo, LICS'21
- Kori, Urabe, Katsumata, Suenaga & Hasuo, CAV'22

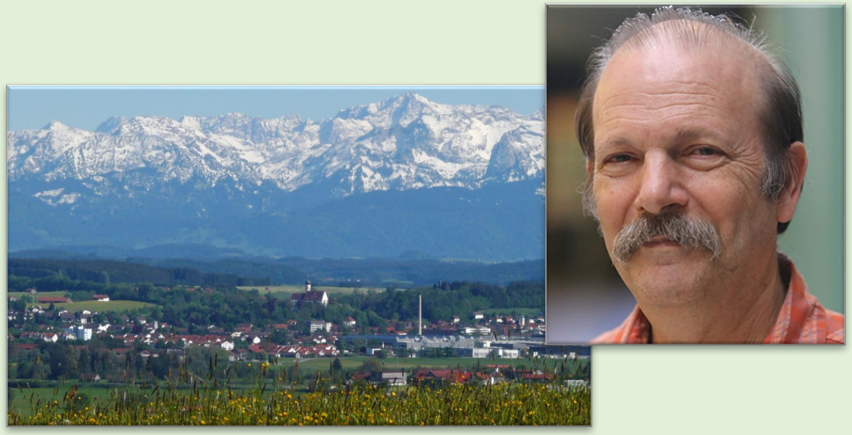
Bridging Categorical Abstract Nonsense and Automata Theory

The categorical community (my background)



- Arrows and diagrams for everything
- *Zhou shall not speak of elements*
- Abstraction, generality

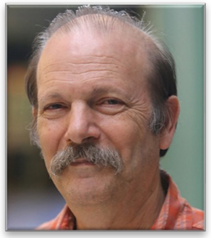
Moshe's model checking lectures at Marktoberdorf 2005



- Moshe: "Non-emptiness of Buechi automata?"
- Audience: "Linear-time!"

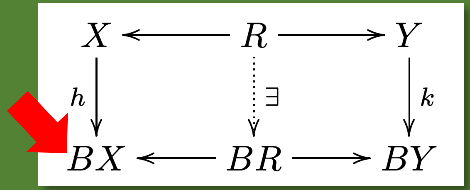
Beauty of computer science:
mathematical elegance at work

Categorical Uniform Definition of Bisimilarity Notions [Rutten, Jacobs, ..., ‘00s]



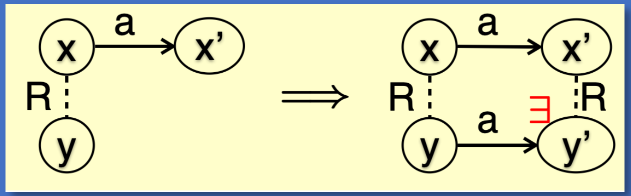
“I like Rutten’s characterization of bisimilarity” (Marktoberdorf ‘05)

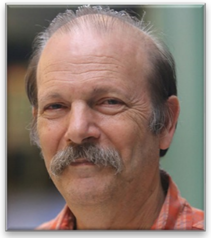
Coalgebraic bisimulation



Categorical abstraction

(Original) bisimulation





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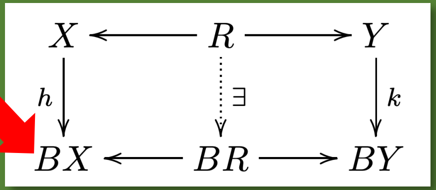


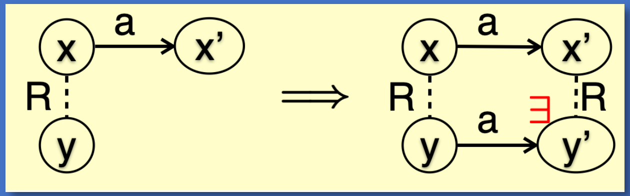
Categorical abstraction

Instantiation

Choosing the parameter B

...

(Original) bisimulation



=

Instance 1 for LTS ($B = P$)

Instance 2 for Markov chains ($B = D$)

II

Probabilistic bisimulation [Larsen & Skou ’91]

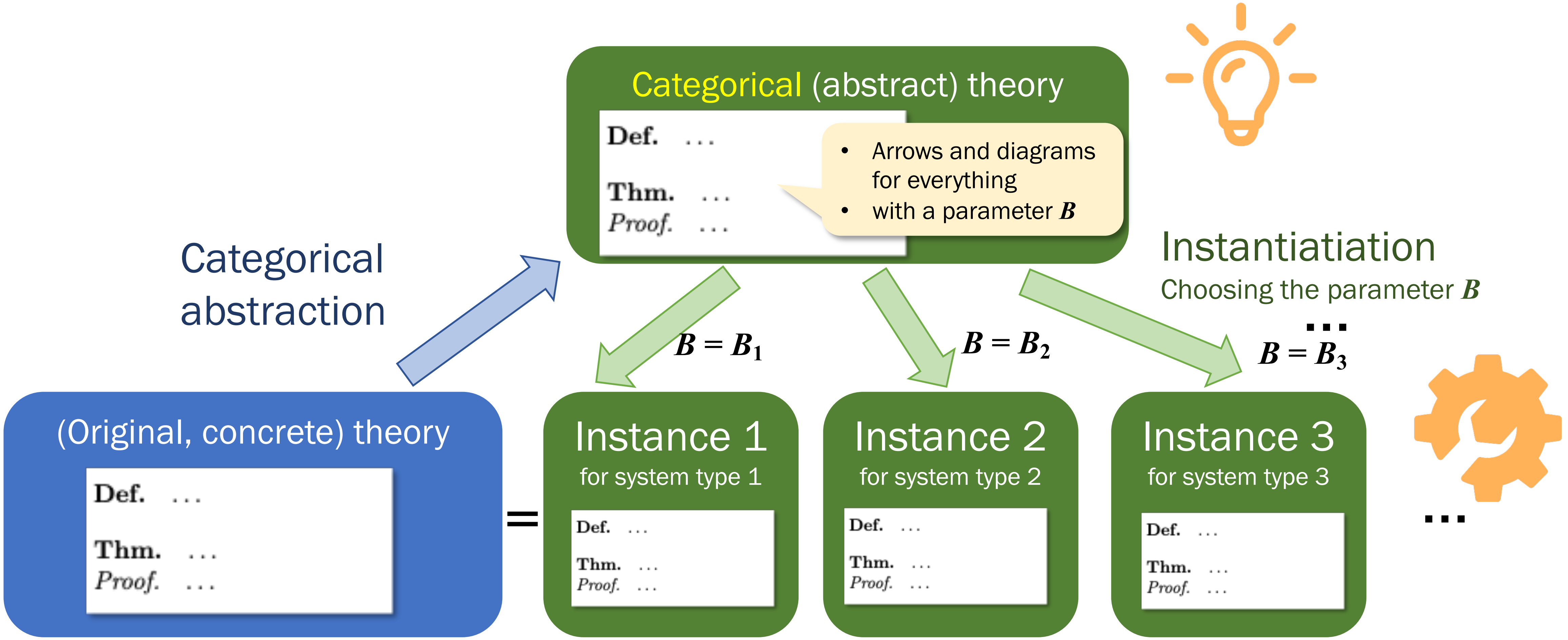
Instance 3 for weighted automata ($B = M_W$)

↓

new bisim. notion

...

Abstraction → Understanding Essences
Instantiation → New Definitions and Theorems



Some instances are known... and some are new

(quantitative/probabilistic, nominal, quantum, ...)

Category Theory is a *Theoretical Backend* that Many Non-Theoreticians Fail to Appreciate

Categorical abstraction

Instantiation

Choosing the parameter B

↑ metatheory

$B = B_1$ $B = B_2$ $B = B_3$

(Original, concrete) theory

Def. …

Thm. …

Proof. …

=

Instance 1

for system type 1

Def. …

Thm. …

Proof. …

Instance 2

for system type 2

Def. …

Thm. …

Proof. …

Instance 3

for system type 3

Def. …

Thm. …

Proof. …

…

↓ (obj.-level) theory

Happy w/ the concrete theories (= frontend).

Why should I care about the backend...

Break the *Fourth Wall*,
Bridge the Object Level and the Meta Level

https://en.wikipedia.org/wiki/Fourth_wall#/media/File:Chicago_Auditorium_Building,_interior_from_balcony.jpg

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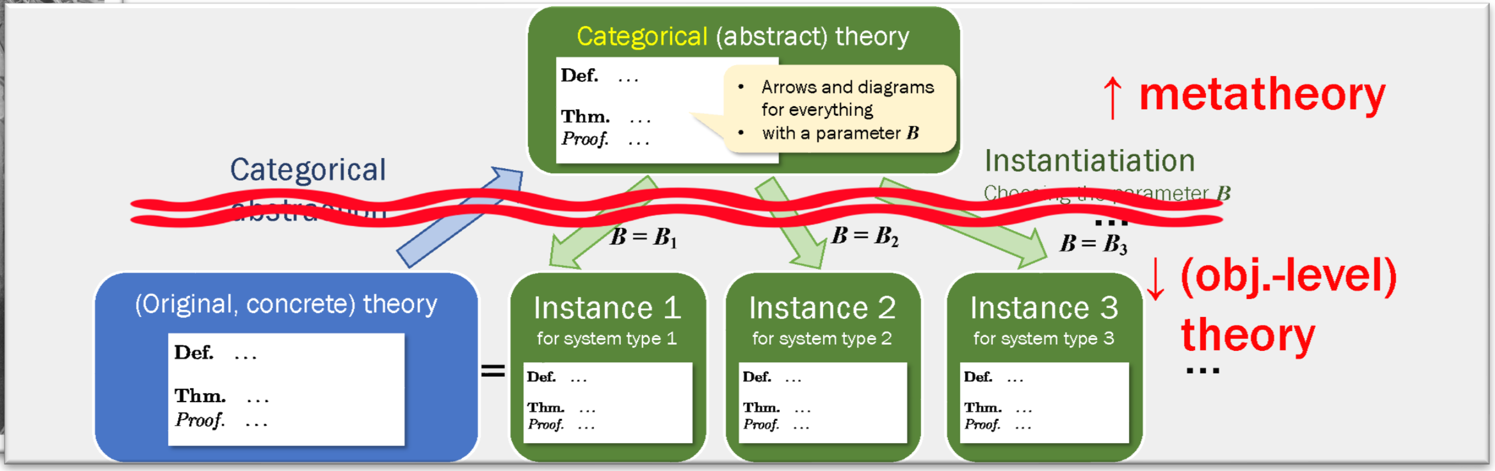
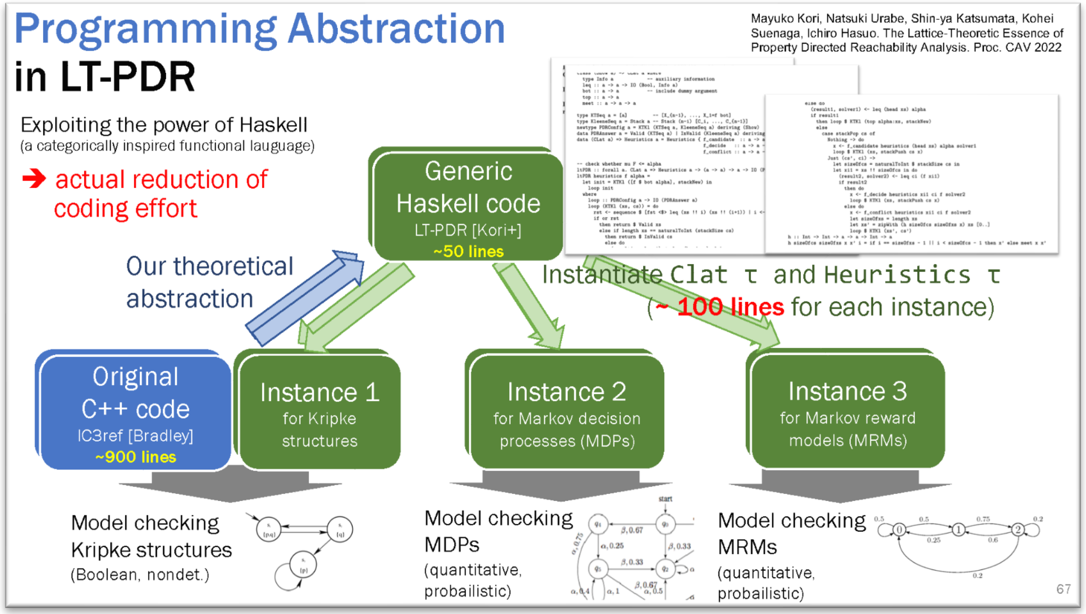
House of Cards



Breaking the Fourth Wall

Line 1: [Kori+, CAV'22]

From mathematical abstraction to programming abstraction



- We can **literally code the abstract theory** thanks to Haskell
- Appl. to IC3/PDR (Bradley, Een, ...): 50 LOC (general) + ~100 LOC each (instant.)
  - vs. original IC3 impl., ~900 LOC in C++

➜ Come to Mayuko's talk, Mon 7 Aug [Kori+, CAV'22]

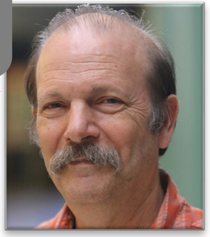
Line 2: [Komorida+, LICS'19] [Komorida+, LICS'21]

Games played in categories— *codensity games*

position	player	possible moves
$P \in \mathbb{E}_X$	Spoiler	$k \in \mathbb{C}(X,\Omega)$ s.t. $\tau \circ Fk \circ c : (X,P) \not\to (\Omega,\Omega)$
$k \in \mathbb{C}(X,\Omega)$	Duplicator	$P' \in \mathbb{E}_X$ s.t. $k : (X,P') \not\to (\Omega,\Omega)$

Moves are inhabitants of categories!
- objects $P \in \mathbb{E}_X$
- arrows $X \xrightarrow{k} \Omega$

"Whatever you do, I can do better with automata"



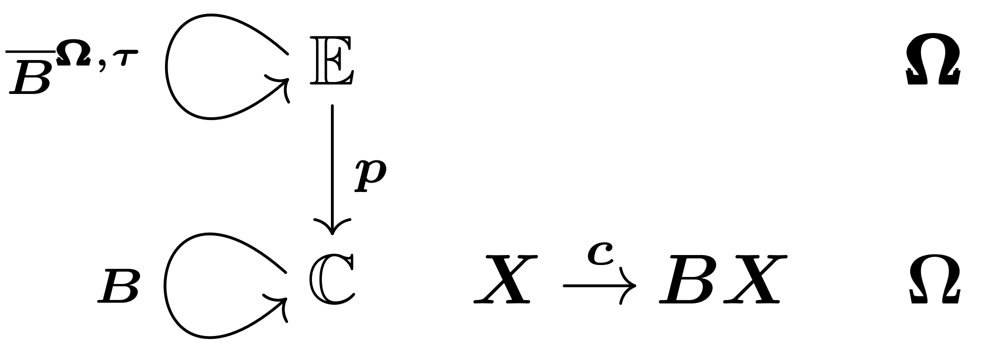
- A concrete technique (namely game characterization) employed at an abstract level
- Demonstrating **the power of automata and games** characterizing fixed points

➜ Rest of this talk

Even More General Definition of Bisimilarity and Its Game Characterization

[Komorida+, LICS'19]
[Komorida+, LICS'21] ➜ modal logic

Setting, Parameters



- Coalgebra [Rutten, Jacobs, ...] + fibration [Benabou, Jacobs, ...]

Def. (codensity lifting)

$$\overline{B}^{\Omega,\tau} P = \bigsqcap_{k \in \mathbb{E}(P,\Omega)} \big(\tau \circ B\big(p(k)\big)\big)^* \Omega$$

Def. (codensity bisimilarity)

$$\nu\left(c^* \circ \overline{B}^{\Omega,\tau}\right), \quad \text{where } \mathbb{E}_X \xrightarrow{\overline{B}^{\Omega,\tau}} \mathbb{E}_{BX} \xrightarrow{c^*} \mathbb{E}_X$$

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Thm. (correctness)
Duplicator is winning at P if and only if $P \sqsubseteq \nu\left(c^* \circ \overline{B}^{\Omega,\tau}\right)$

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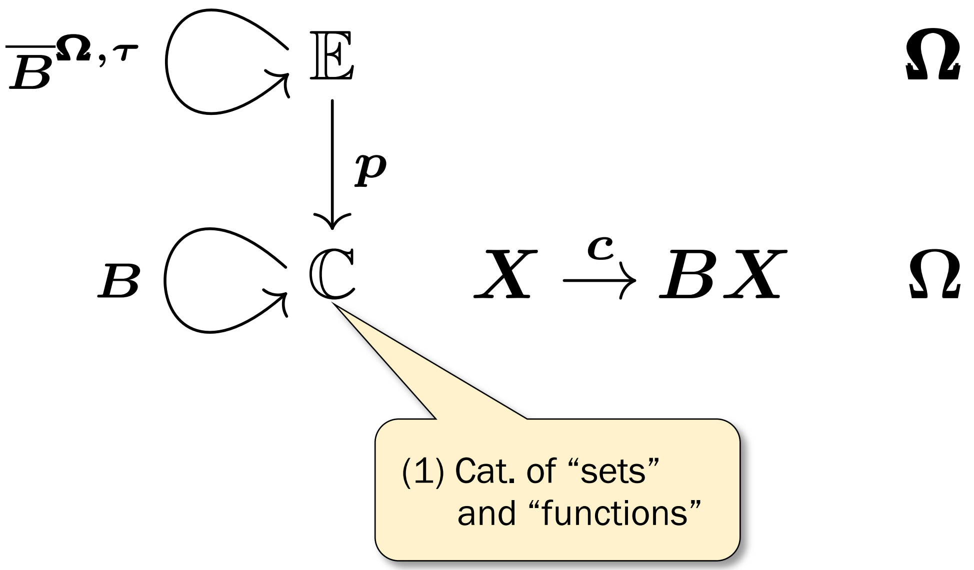

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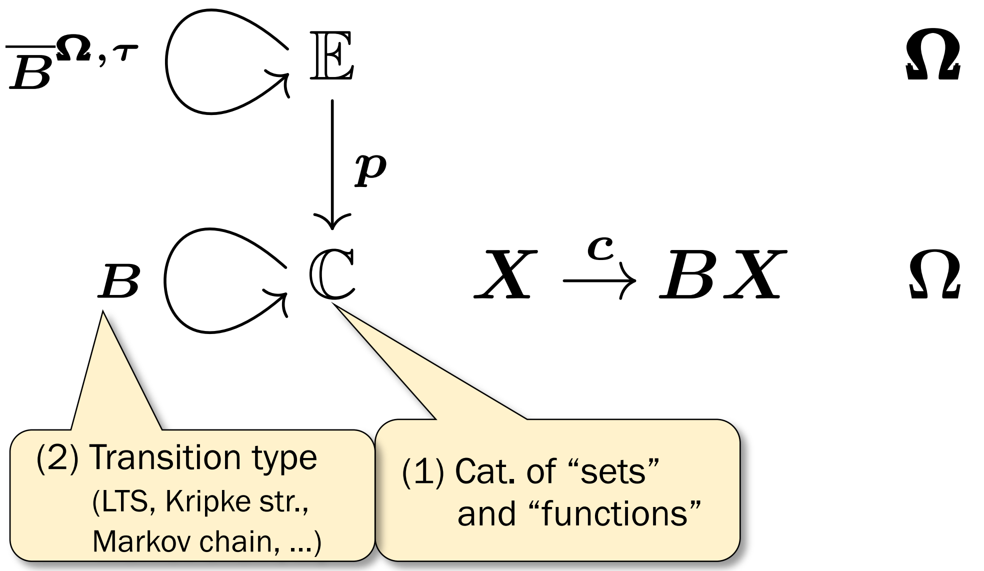

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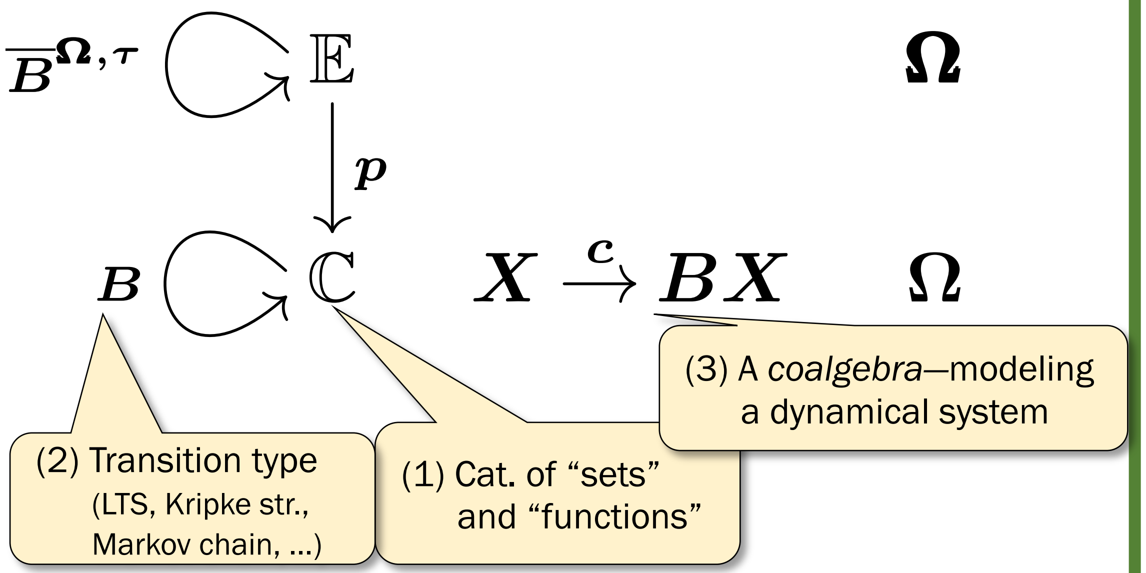

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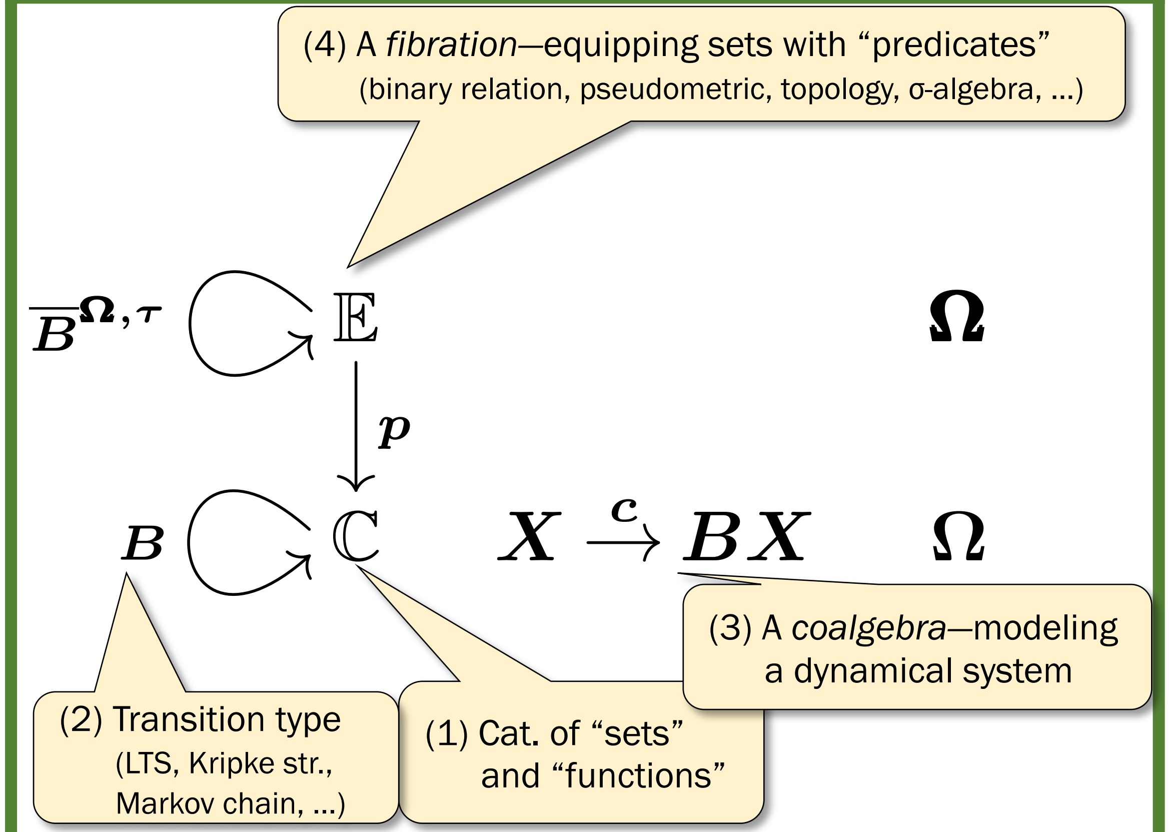

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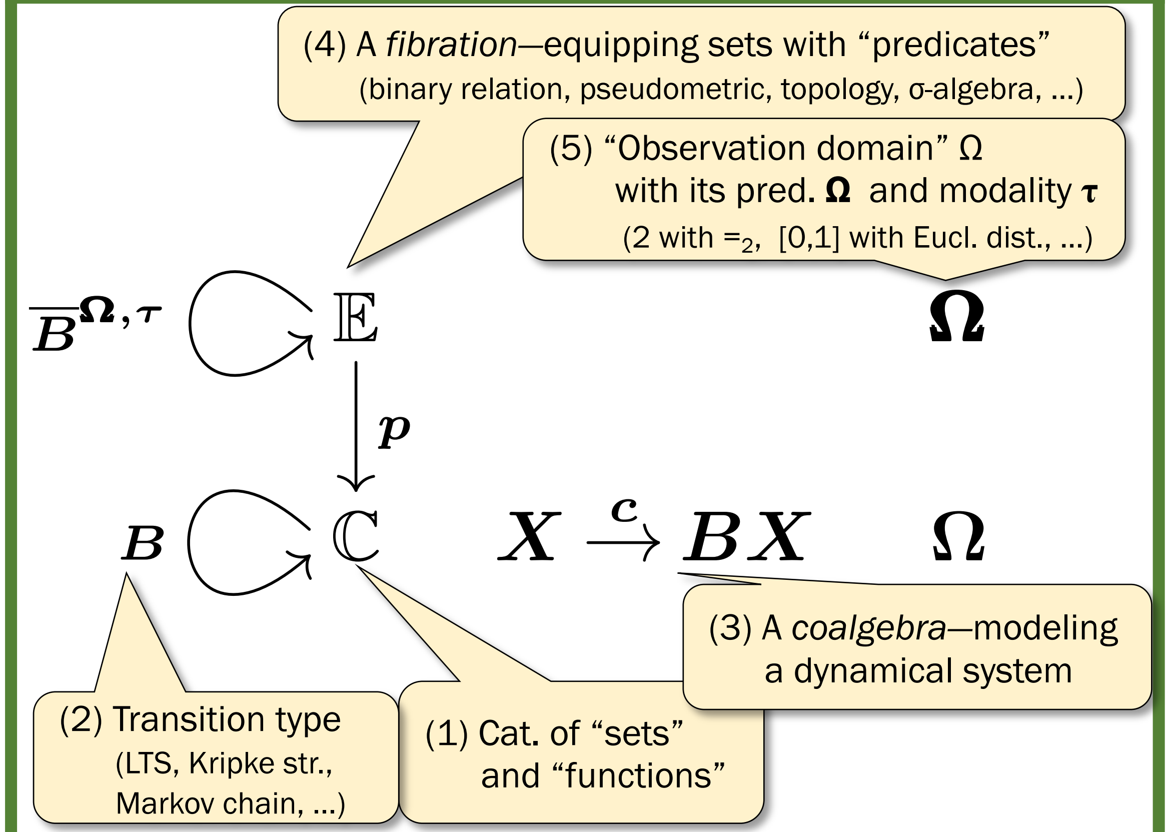

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A Variety of Bisimulation Games—Qualitative and Quantitative Alike

Games for codensity bisimilarity

Played in categories!

position	player	possible moves
$P \in \mathbb{E}_X$	Spoiler	$k \in \mathbb{C}(X, \Omega)$ s.t. $\tau \circ Fk \circ c : (X, P) \dot{\nrightarrow} (\Omega, \Omega)$
$k \in \mathbb{C}(X, \Omega)$	Duplicator	$P' \in \mathbb{E}_X$ s.t. $k : (X, P') \dot{\nrightarrow} (\Omega, \Omega)$

Categorical abstraction

Instantiation

Choosing the parameters B, Ω, $\mathbb{E}$, τ, …

…

Game for prob. bisim.

[Fijalkow+ ICALP'17]

position	player	possible moves
$(x, y) \in X^2$	Spoiler	$Z \subseteq X$ s.t. $c(x)(Z) \neq c(y)(Z)$
$Z \subseteq X$	Duplicator	$(x', y') \in X^2$ s.t. $x' \in Z \wedge y' \notin Z$

Conventional bisim. (Kripke frames)

position	player	possible moves
$(x, y) \in X \times X$	Spoiler	$k \in \mathbf{Set}(X, 2)$ such that exactly one of $\exists x' \in c(x).\ k(x') = \top$ and $\exists y' \in c(y).\ k(y') = \top$ holds
$k \in \mathbf{Set}(X, 2)$	Duplicator	(x'', y'') s.t. $k(x'') \neq k(y'')$

Different game from the standard "mimicking" game

Bisimulation metric

position	player	possible moves
$(x, y, \varepsilon) \in X^2 \times [0, 1]$	Spoiler	$f : X \to [0, 1]$ such that $\lvert E_{c(x)}[f] - E_{c(y)}[f] \rvert > \varepsilon$
$f : X \to [0, 1]$	Duplicator	$(x', y', \varepsilon') \in X^2 \times [0, 1]$ such that $\lvert f(x') - f(y') \rvert > \varepsilon'$

Different game from [Koenig+, CONCUR'18] that is Wasserstein-based

"Bisimulation topology"

position	player	possible moves
$\mathcal{O} \in \mathbf{Top}_X$	Spoiler	$a \in \{\varepsilon\} \cup \Sigma$ and $k \in \mathbf{Set}(X, 2)$ such that $\tau_a \circ (A_\Sigma k) \circ c : X \to 2$ is not continuous from $(X, \mathcal{O})$ to $(2, \Omega_a)$
$a \in \{\varepsilon\} \cup \Sigma$ and $k \in \mathbf{Set}(X, 2)$	Duplicator	$\mathcal{O}' \in \mathbf{Top}_X$ such that $k : X \to 2$ is not continuous from $(X, \mathcal{O}')$ to $(2, \Omega_a)$

…

Abstract Yet Intuitive Bisimulation Notion and Game Characterization

Def. (codensity lifting)

$$\overline{B}^{\boldsymbol{\Omega},\tau} P = \bigsqcap_{k \in \mathbb{E}(P,\boldsymbol{\Omega})} \big(\tau \circ B\big(p(k)\big)\big)^{*} \boldsymbol{\Omega}$$

... carries

- a pred. P over X (the current state)
- to one over BX (the next state)

$$\begin{array}{lccc} \mathbb{E} & P & \longmapsto & \overline{B}^{\boldsymbol{\Omega},\tau}(P) \\ \downarrow p & & & \\ \mathbb{C} & X & \xrightarrow{\ c\ } & BX \end{array}$$

by

$$\begin{array}{lccccc} \mathbb{E} & (\tau \circ Bk)^{*}\boldsymbol{\Omega} & \cdots\cdots\cdots\cdots\rightarrow & & & \boldsymbol{\Omega} \\ \downarrow p & & & & & \\ \mathbb{C} & BX & \xrightarrow{\ Bk\ } & B\Omega & \xrightarrow{\ \tau\ } & \Omega. \end{array}$$

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- Conventional bisim. games: **challenge-defend**
- Codensity games: **blame-blame**

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by

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(1) *Observation* $k : X \to \Omega$ made in the next state ($Bk : BX \to B\Omega$)

(2) Collapsed by *modality* $\tau : B\Omega \to \Omega$

(3) ... along which the obs. pred. Ω is pulled back

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ERATO MMSD

Abstract Yet Intuitive Bisimulation Notion and Game Characterization

(4) ... for all obs. $k : X \to \Omega$ that *respects* P (relation-preserving, non-expansive, continuous, ...)

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position	player	…sible moves
$P \in \mathbb{E}_X$	Spoiler	$k \in \mathbb{C}(X,\Omega)$ s.t. $\tau \circ Bk \circ c : (X,P) \dot{\nrightarrow} (\Omega,\mathbf{\Omega})$
$k \in \mathbb{C}(X,\Omega)$	Duplicator	$P' \in \mathbb{E}_X$ s.t. $k : (X,P') \dot{\nrightarrow} (\Omega,\mathbf{\Omega})$

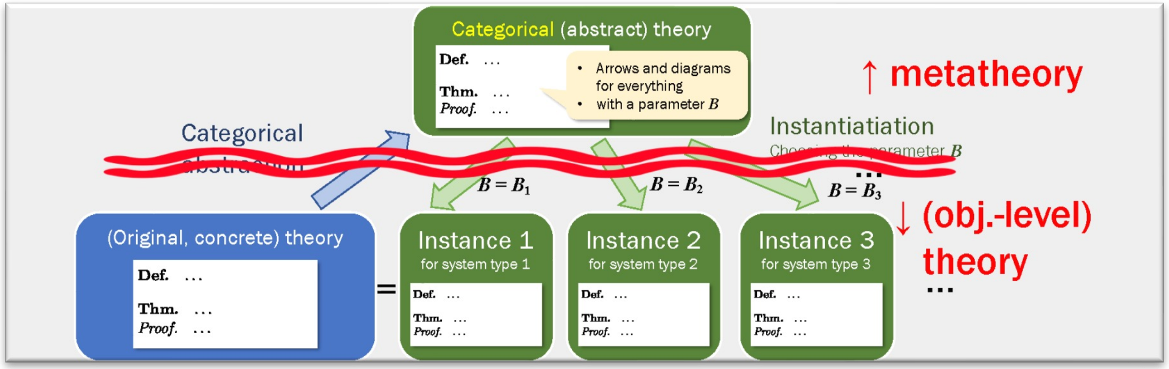
You're lying...
by claiming P as a bisimulation
which is coarser than obs. $k : X \to \Omega$

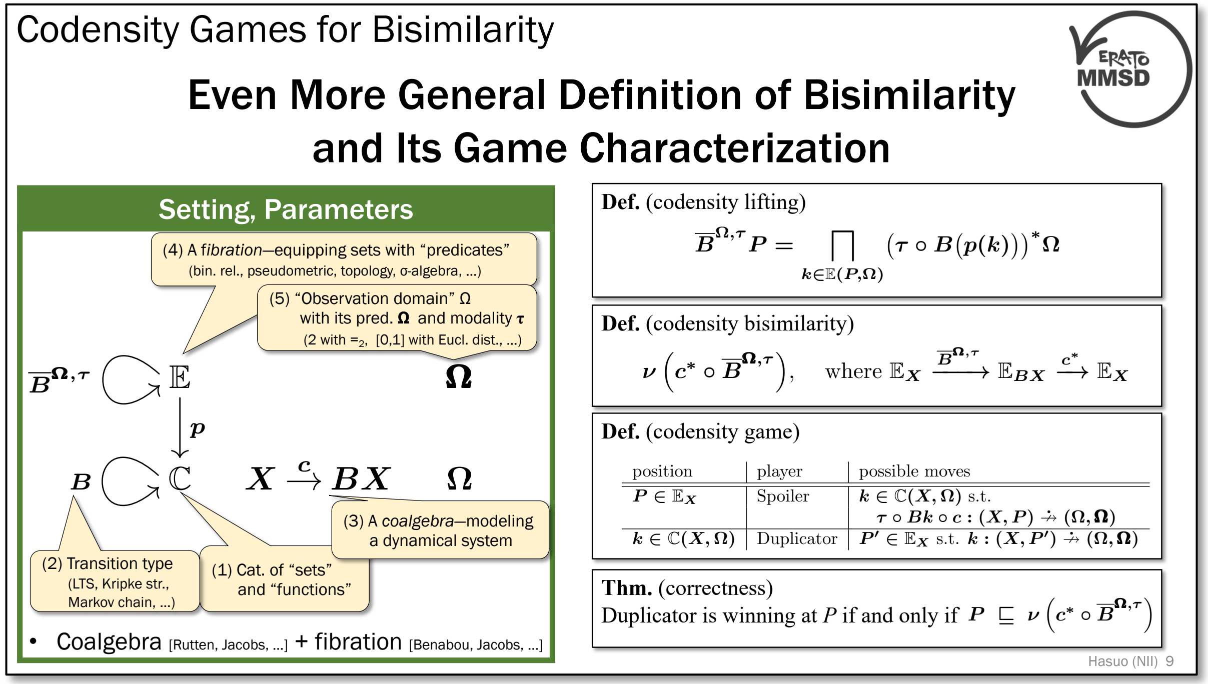
You're lying...
by using obs. $k : X \to \Omega$
that is illegitimately fine-grained

- Conventional bisim. games: **challenge-defend**
- Codensity games: **blame-blame**

Conclusions

Bridging Categorical Abstract Nonsense and Automata Theory





Break the *Fourth Wall*,
Bridge the Object Level and the Meta Level

Coalgebra + Fibration
→ General Bisimulation Game in Categories

… and thanks a million, Moshe, for your inspirations!